Compact Gradient Shrinking Ricci Solitons
with Positive Curvature Operator

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February 2, 2008

Abstract

In this paper, we first derive several identities on a compact shrinking Ricci soliton. We then show that a compact gradient shrinking soliton must be Einstein, if it admits a Riemannian metric with positive curvature operator and satisfies an integral inequality. Furthermore, such a soliton must be of constant curvature.

1 Introduction and Main Theorems

Hamilton started the study of the Ricci flow in [2]. In [3], Hamilton has classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman made significant progress in his recent work [5] and [6].

Suppose we have a solution to the Ricci flow

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij} \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric $g(t)$. Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. If the vector field which induce the diffeomorphism is in fact the gradient of a function, we call it a gradient Ricci soliton. For a gradient shrinking Ricci soliton, we have the equation

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2\tau} g_{ij} \, , \tag{1.2}$$

1

where $\tau = T - t$. T is the time the soliton becomes a point, and f is called Ricci potential function. In the special case when f is a constant, then we have an Einstein manifold.

Besides the above equation, a gradient shrinking Ricci soliton must also satisfies the following equations,

$$R + \triangle f = \frac{n}{2\tau} \qquad (1.3)$$

and

$$R + |\nabla f|^2 = \frac{f - c}{\tau} , \qquad (1.4)$$

where c is a constant in space. The last equation (1.4) determines the value of f. The Ricci potential function f satisfies the following evolution equation,

$$\frac{\partial}{\partial t} f = |\nabla f|^2 . \qquad (1.5)$$

Inspired by his own work in [3] and [4], Hamilton made the following conjecture:

Conjecture 1. (**Hamilton**) *A compact gradient shrinking Ricci soliton with positive curvature operator must be Einstein.*

On the other hand, it is a well-known theorem of Tachibana [8] that any compact Einstein manifold with positive sectional curvature must be of constant curvature. Hence Hamilton's conjecture is a generalization of the Tachibana theorem, since Einstein manifolds are special Ricci solitons with constant Ricci potential functions.

In this paper, we first derive a sequence of identities on gradient shrinking Ricci solitons. Then we show that the above conjecture is in fact true provided that the Ricci soliton satisfies an integral inequality.

One of our main theorems is the following:

Theorem 1. *Let $(M, g(t))$ be a compact gradient shrinking Ricci soliton, then M must be of constant curvature if its curvature operator is positive and satisfies the following inequality,*

$$\frac{1}{2} \int |Rc|^2 |\nabla f|^2 e^{-f} \leq \int K e^{-f} + \int R_{ijkl} R_{ik} f_j f_l e^{-f} , \qquad (1.6)$$

where

$$K = (\nabla_i \nabla_j R_{ik} - \nabla_j \nabla_i R_{ik}) R_{jk} . \qquad (1.7)$$

In Section Two, we first derive some integral identities about Riemannian curvature on gradient shrinking Ricci solitons. More precisely, we prove the following two identities,

2

Theorem 2. *On a compact gradient shrinking Ricci soliton, we have*

$$\int Rm(Rc, Rc)e^{-f} = \frac{1}{2\tau}\int |Rc|^2 e^{-f} + \frac{1}{2}\int |div\ Rm|^2 e^{-f} \qquad (1.8)$$

and

$$\int Rm(Rc, Rc)e^{-f} = \frac{1}{2\tau}\int |Rc|^2 e^{-f} + \int |\nabla Rc|^2 e^{-f} - \frac{1}{2}\int |div\ Rm|^2 e^{-f}\ , \qquad (1.9)$$

where

$$Rm(Rc, Rc) = R_{ijkl}R_{ik}R_{jl}\ . \qquad (1.10)$$

As a corollary of Theorem 2, we have

Corollary 1. *On a compact gradient shrinking Ricci soliton, we have*

$$\int |\nabla Rc|^2 e^{-f} = \int |div\ Rm|^2 e^{-f}\ . \qquad (1.11)$$

Moreover, (1.8) and (1.9) can be written as follows,

$$\int Rm(Rc, Rc)e^{-f} = \frac{1}{2\tau}\int |Rc|^2 e^{-f} + \frac{1}{2}\int |\nabla Rc|^2 e^{-f}\ . \qquad (1.12)$$

In Section Three, we derive some identities about Ricci curvature, i.e., we show the following theorem,

Theorem 3. *On a compact gradient shrinking Ricci soliton, we have*

$$\frac{1}{2}\int |Rc|^2 \triangle(e^{-f}) = \frac{1}{2}\int |\nabla Rc|^2 e^{-f} + \int Ke^{-f} + \int R_{kljp}R_{kj}f_l f_p e^{-f}\ . \qquad (1.13)$$

In Section Four, we prove Theorem 1 under the hypothesis of positive curvature operator and inequality (1.6).

2 Identities of Riemannian Curvature

In this section, we prove Theorem 2. On a gradient shrinking Ricci soliton, we have the following identities:

$$\begin{aligned}
(div\ Rm)_{jkl} &= R_{ijkl,i} = \nabla_i R_{ijkl} = \nabla_i R_{klij} \\
&= -\nabla_k R_{ijli} - \nabla_l R_{ijik} = \nabla_k R_{jl} - \nabla_l R_{jk} \\
&= -\nabla_k f_{jl} + \nabla_l f_{jk} = \nabla_l \nabla_k f_j - \nabla_k \nabla_l f_j \\
&= R_{lkjp} f_p \ .
\end{aligned} \tag{2.1}$$

Hence we have the following two identities,

$$\nabla_i(R_{ijkl} e^{-f}) = 0 \tag{2.2}$$

and

$$\nabla_i(R_{ik} e^{-f}) = 0 \ . \tag{2.3}$$

Using integration by parts, we derive that

$$\begin{aligned}
&\int |div\ R_m|^2 e^{-f} \\
={}& \int R_{lkjp} f_p (-R_{jk,l} + R_{jl,k}) e^{-f} \\
={}& \int R_{lkjp} f_p R_{jl,k} e^{-f} - \int R_{lkjp} f_p R_{jk,l} e^{-f} \\
={}& -\int R_{lkjp} f_{pk} R_{jl} e^{-f} + \int R_{lkjp} f_{pl} R_{jk} e^{-f} \\
={}& -\int R_{lkjp} R_{lj} f_{kp} e^{-f} - \int R_{kljp} R_{kj} f_{lp} e^{-f} \\
={}& -2 \int R_{lkjp} R_{lj} f_{kp} e^{-f} \ .
\end{aligned}$$

Hence we have the following lemma:

Lemma 1. *On a gradient shrinking Ricci soliton, we have*

$$\int R_{lkjp} R_{lj} f_{kp} e^{-f} = -\frac{1}{2} \int |div\ Rm|^2 e^{-f} \leq 0 \ . \tag{2.4}$$

Now we can prove (1.8) in Theorem 2.

Proof. By the above lemma and the gradient shrinking Ricci soliton equation:

$$f_{kp} = \frac{1}{2\tau} g_{kp} - R_{kp} \ ,$$

4

we can derive

$$\int |div\ R_m|^2 e^{-f}$$

$$= -2\int R_{lkjp}R_{lj}(\frac{1}{2\tau}g_{kp} - R_{kp})e^{-f}$$

$$= -\frac{1}{\tau}\int |Rc|^2 e^{-f} + 2\int Rm(Rc, Rc)e^{-f}\ ,$$

so we have

$$\int Rm(Rc, Rc)e^{-f} = \frac{1}{2\tau}\int |Rc|^2 e^{-f} + \frac{1}{2}\int |div\ Rm|^2 e^{-f}\ .$$

$\square$

Before we prove (1.9), we first prove the following two lemmas:

Lemma 2.
$$\nabla_i\nabla_j R_{ik} - \nabla_j\nabla_i R_{ik} = R_{jm}R_{mk} - R_{ijmk}R_{im} \tag{2.5}$$

Proof. Using the formula

$$\nabla_i\nabla_j R_{lk} - \nabla_j\nabla_i R_{lk} = -R_{ijml}R_{mk} - R_{ijmk}R_{lm}\ ,$$

and let $i = l$ in the above formula and take the sum. $\square$

Lemma 3. *On a gradient shrinking Ricci soliton,*

$$-2\int \nabla_k R_{jl}\nabla_l R_{jk}e^{-f} = \frac{1}{\tau}\int |Rc|^2 e^{-f} - 2\int Rm(Rc, Rc)e^{-f}\ . \tag{2.6}$$

Proof.

$$-2\int \nabla_k R_{jl}\nabla_l R_{jk}e^{-f}$$

$$=2\int R_{jk}(\nabla_i\nabla_j R_{ik} - \nabla_j R_{ik}f_i)e^{-f}$$

$$=2\int R_{jk}(\nabla_i\nabla_j R_{ik})e^{-f} - 2\int R_{jk}\nabla_j R_{ik}f_i e^{-f}$$

$$=2\int R_{jk}(\nabla_j\nabla_i R_{ik} + R_{mj}R_{mk} - R_{ijmk}R_{im})e^{-f} + 2\int R_{ik}R_{jk}f_{ij}e^{-f}$$

$$=0 + 2\int R_{jk}R_{mj}R_{mk}e^{-f} + 2\int R_{ik}R_{jk}f_{ij}e^{-f} - 2\int R_{ijmk}R_{im}R_{jk}e^{-f}$$

$$=2\int R_{jk}R_{ki}(f_{ij} + R_{ij})e^{-f} - 2\int R_{ijmk}R_{im}R_{jk}e^{-f}$$

$$=\frac{1}{\tau}\int |Rc|^2 e^{-f} - 2\int Rm(Rc, Rc)e^{-f}\ .$$

This finishes the proof of the lemma. $\square$

5

We used the following lemma in the above,

Lemma 4. *On a gradient shrinking Ricci soliton, we have*

$$\int \nabla_j \nabla_i R_{ik} R_{jk} e^{-f} = 0 \ . \tag{2.7}$$

Proof.

$$\int \nabla_j \nabla_i R_{ik} R_{jk} e^{-f} = -\int \nabla_i R_{ik} \nabla_j (R_{jk} e^{-f}) = 0 \ .$$

□

Now we can prove (1.9) in Theorem 2.

Proof.

$$\int |div \ R_m|^2 e^{-f}$$

$$= \int |\nabla_k R_{jl} - \nabla_l R_{jk}|^2 e^{-f}$$

$$= 2 \int |\nabla Rc|^2 e^{-f} - 2 \int \nabla_k R_{jl} \nabla_l R_{jk} e^{-f}$$

$$= 2 \int |\nabla Rc|^2 e^{-f} + \frac{1}{\tau} \int |Rc|^2 e^{-f} - 2 \int Rm(Rc, Rc) e^{-f} \ ,$$

so

$$\int Rm(Rc, Rc) e^{-f} = \frac{1}{2\tau} \int |Rc|^2 e^{-f} + \int |\nabla Rc|^2 e^{-f} - \frac{1}{2} \int |div \ R_m|^2 e^{-f} \ .$$

□

By (1.8) and (1.9) we have the corollary 1.

3   Identities of Ricci Curvature

Because of the soliton equation, there will be several identities for Ricci curvature on the gradient shrinking Ricci solitons. We first prove Theorem 3. By using (2.1), we derive that

$$\Delta R_{jk} = \nabla_i(\nabla_i R_{jk}) = \nabla_i(\nabla_j R_{ik} - R_{ijkl} f_l) = \nabla_i \nabla_j R_{ik} - (\nabla_i R_{ijkl}) f_l - R_{ijkl} f_{li} \ , \tag{3.1}$$

so

$$< \Delta Rc, Rc >= \nabla_i \nabla_i R_{jk} R_{jk} = \nabla_i \nabla_j R_{ik} R_{jk} - (\nabla_i R_{ijkl}) f_l R_{jk} - R_{ijkl} f_{li} R_{jk} \ , \tag{3.2}$$

and

$$\frac{1}{2}\Delta|Rc|^2 = \frac{1}{2}\Delta(R_{jk}R_{jk}) = \nabla_i(\nabla_i R_{jk}R_{jk}) = (\Delta R_{jk}R_{jk}) + |\nabla Rc|^2 . \qquad (3.3)$$

Furthermore, we have

$$\frac{1}{2}\int \Delta|Rc|^2 e^{-f} = \frac{1}{2}\int |Rc|^2 \Delta e^{-f}$$

so

$$\frac{1}{2}\int |Rc|^2 \Delta e^{-f}$$

$$= \int <\Delta Rc, Rc> e^{-f} + \int |\nabla Rc|^2 e^{-f}$$

$$= \int |\nabla Rc|^2 e^{-f} + \int (\nabla_i \nabla_j R_{ik}R_{jk} - \nabla_j \nabla_i R_{ik}R_{jk})e^{-f}$$

$$+ \int \nabla_j \nabla_i R_{ik}R_{jk}e^{-f} - \int \nabla_i R_{ijkl}f_l R_{jk}e^{-f} - \int R_{ijkl}f_{li}R_{jk}e^{-f}$$

$$= \int |\nabla Rc|^2 e^{-f} + \int Ke^{-f} + \int \nabla_j \nabla_i R_{ik}R_{jk}e^{-f} - \int \nabla_i R_{ijkl}f_l R_{jk}e^{-f} - \int R_{ijkl}f_{li}R_{jk}e^{-f}$$

$$= \int |\nabla Rc|^2 e^{-f} + \int Ke^{-f} - \int \nabla_i R_{ijkl}f_l R_{jk}e^{-f} - \int R_{ijkl}f_{li}R_{jk}e^{-f} . \qquad (3.4)$$

We used Lemma 4 in the last equation.

Plug (2.1) and (2.4) into (3.4), apply Corollary 1, we obtain

$$\frac{1}{2}\int |Rc|^2 \Delta e^{-f} = \int |\nabla Rc|^2 e^{-f} + \int Ke^{-f} - \int \nabla_i R_{ijkl}f_l R_{jk}e^{-f} - \frac{1}{2}\int |\nabla Rc|^2 e^{-f}$$

$$= \frac{1}{2}\int |\nabla Rc|^2 e^{-f} + \int Ke^{-f} + \int R_{kljp}R_{jk}f_l f_p e^{-f} . \qquad (3.5)$$

If we assume that the metric on the gradient shrinking Ricci soliton has positive curvature, then

$$\int R_{kljp}R_{jk}f_p f_{li}e^{-f}$$

is a positive term. In fact, this is true for any metric with positive curvature operator. We have

Lemma 5. *Let (M,g) be a Riemannian manifold with positive curvature operator, then*

$$R_{ijkl}R_{ik}f_j f_l \geq 0$$

point-wise.

Proof.

$$R_{ikjl} = \sum_\alpha \lambda_\alpha \omega_{ik}^\alpha \omega_{jl}^\alpha \ ,$$

where

$$\omega^\alpha = \omega_{ik}^\alpha dx^i \wedge dx^k$$

are 2-forms (in fact, they are the eigenfunctions of the curvature operator). And

$$\lambda_\alpha \geq 0 \ ,$$

so

$$R_{ikjl} R_{ij} f_k f_l = \sum_\alpha \lambda_\alpha [\omega_{ik}^\alpha \omega_{jl}^\alpha R_{ij} f_k f_l] \ ,$$

with

$$\omega_{ik}^\alpha \omega_{jl}^\alpha R_{ij} f_k f_l = R_{ij}(\omega_{ik}^\alpha f_k)(\omega_{jl}^\alpha f_l) = R_{ij} \gamma_i^\alpha \gamma_j^\alpha \geq 0$$

and

$$\gamma_i^\alpha = \omega_{ik}^\alpha f_k \ .$$

$\square$

Remark. It's an easy calculation to see that

$$\frac{1}{2} \int |Rc|^2 \Delta e^{-f} = \frac{1}{\tau} \int Rc(\nabla f, \nabla f) e^{-f} \geq 0 \ .$$

4 Proof of Theorem 1

For a compact Riemannian manifold with positive curvature operator, we first need the following lemma of Berger:

Lemma 6. *(**Berger**) Assume T is a symmetric two tensor on a Riemannian manifold (M, g) with non-negative sectional curvature, then*

$$K = (\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} \geq 0 \ .$$

In fact,

$$K = \sum_{i<j} R_{ijij}(\lambda_i - \lambda_j)^2 \ ,$$

where λ_i's are the eigenvalues of T.

We apply this lemma in the special case of

$$T = Rc .$$

Then we know that our K which is defined in (1.7) is non-negative.

By combining Theorem 3 and inequality (1.6), we can prove Theorem 1.

Proof. By

$$\frac{1}{2} \int |Rc|^2 \triangle (e^{-f})$$

$$= \int \nabla_i R_{jk} R_{jk} f_i e^{-f}$$

$$\leq \frac{1}{2} \int |\nabla Rc|^2 e^{-f} + \frac{1}{2} \int |Rc|^2 |\nabla f|^2 e^{-f}$$

$$\leq \frac{1}{2} \int |\nabla Rc|^2 e^{-f} + \int K e^{-f} + \int R_{ijkl} R_{ik} f_j f_l e^{-f} ,$$

we show that for all i, j and k we have

$$\nabla_i R_{jk} = R_{jk} f_i ,$$

and

$$\int R_{kljp} R_{kj} f_l f_p e^{-f} = \int R_{kljp} \nabla_l R_{kj} f_p e^{-f} = - \int R_{kljp} R_{kj} f_{lp} e^{-f} = \frac{1}{2} \int |\nabla Rc|^2 e^{-f} .$$

So

$$\int K e^{-f} = 0 ,$$

hence

$$K \equiv 0$$

and f is a constant. Therefore, the soliton must be of constant curvature. $\square$

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